1 Introduction

Let’s begin with the simplest question: what, exactly, is a join? If you remember the SQL project, you’ll remember writing things like `R INNER JOIN S ON R.name = S.name` and other similar statements.

What that actually meant is that you take two relations, R and S, and create one new relation out of their matches on the join condition – that is, for each record \( r_i \) in R, find all records \( s_j \) in S that match the join condition we have specified and write \( <r_i, s_j> \) as a new row in the table (all the fields of \( r \) followed by all the fields of \( s \)). The SQL lecture slides are a great resource for more clarifications on what joins actually are.\(^1\)

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\(^1\) **notation aside:** \(|T|\) is the number of pages in table T and \( |T| \) is the number of records in table T. This is really essential to understand the following explanations.
2 Simple Nested Loop Join

Let’s start with the simplest strategy possible. Let’s say we have a buffer of B pages, and let’s say that we wish to join two tables, R and S, on the join condition \( \theta \). Starting with the most naïve strategy, we can take each record in R, search for all its matches in S, and then write each match to disk.

This is called simple nested loop join (SNLJ). You can think of it as two nested for loops:

for each record \( r_i \) in R:
    for each record \( s_j \) in S:
        if \( \theta(r_i,s_j) \):
            write \( <r_i,s_j> \) to disk

This would be a great thing to do, but the theme of the class is really centered around optimization and minimizing I/Os. For that, this is a pretty poor scheme, because we take each record in R and read in every single page in S searching for a match. The I/O cost of this would then be \([R] + |R|[S]\), where \([R]\) is the number of pages in R and \(|R|\) is the number of records in R...and while we might be able to optimize things a slight amount by switching the order of R and S in the for loop, this really isn’t a very good strategy.

Note: SNLJ does not incur \(|R|\) I/Os to read every record in R. It will cost \([R]\) I/Os because its really doing something more like “for each page \( p_r \) in R: for each record \( r \) in \( p_r \); for each page \( p_s \) in S: for each record \( s \) in \( p_s \): join” since we can’t read less than a page at a time.
3 Page Nested Loop Join

It’s clear that we don’t want to read in every single page of \( S \) for each record of \( R \), so what can we do better? What if we read in every single page in \( S \) for every single page of \( R \) instead? That is, for a page of \( R \), take all the records and match them against each record in \( S \), and do this for every page of \( R \).

That’s called page nested loop join (PNLJ). Here’s the pseudocode for it:

```plaintext
for each page \( p_r \) in \( R \):
    for each page \( p_s \) in \( S \):
        for each record \( r_i \) in \( p_r \):
            for each record \( s_j \) in \( p_s \):
                if \( \theta(r_i, s_j) \):
                    yield \( <r_i, s_j> \)
```

The I/O cost of this is somewhat better. It’s \([R] + [R][S] \) – this can be optimized by keeping the smaller relation between \( R \) and \( S \) as the outer one.
4 Block Nested Loop Join

Page Nested Loop Join is a lot better! The only problem is that we’re still not fully utilizing our buffer as powerfully as we can. We have $B$ buffer pages, but our algorithm only uses 3 – one for $R$, one for $S$, and one for the output buffer. Remember that the fewer times we read in $S$, the better – so if we can reserve $B-2$ pages for $R$ instead and match $S$ against every record in each ”chunk”, we could cut down our I/O cost drastically!

This is called **Chunk Nested Loop Join (or Block Nested Loop Join)**. The key idea here is that we want to utilize our buffer to help us reduce the I/O cost, and so we can reserve as many pages as possible for a chunk of $R$ – because we only read in each page of $S$ once per chunk, larger chunks imply fewer I/Os. For each chunk of $R$, match all the records in $S$ against all the records in the chunk.

for each block of $B-2$ pages $B_r$ in $R$:
  for each page $p_s$ in $S$:
    for each record $r_i$ in $B_r$:
      for each record $s_j$ in $p_s$:
        if $\theta(r_i, s_j)$:
          yield $<r_i, s_j$>

Then, the I/O cost of this can be written as $|R| + \lceil \frac{|R|}{B-2} \rceil |S|$. This is a lot better! Now, we’re taking advantage of our $B$ buffer pages to reduce the number of times we have to read in $S$. 

![Block Nested Loop Join Diagram](image-url)
5 Index Nested Loop Join

There are times, however, when Block Nested Loop Join isn’t the best thing to do. Sometimes, if we have an index on $S$ that is on the appropriate field (i.e. the field we are joining on), it can be very fast to look up matches of $r_i$ in $S$. This is called index nested loop join, and the pseudocode goes like this:

```
for each record $r_i$ in R:
    for each record $s_j$ in S where $\theta(r_i, s_j) == true$:
        yield $<r_i, s_j>$
```

The I/O cost is $|R| + |R| \times $ (cost to look up matching records in $S$).

The cost to look up matching records in $S$ will differ based on the type of index. If it is a B+ tree, we will search starting at the root and count how many I/Os it will take to get to a corresponding record. See the Clustering and Counting I/O’s sections of the B+ tree course notes.
6 Hash Join

Notice that in this entire sequence, we’re really trying to look for matching records. Hash tables are really nice for looking up matches, though; even if we don’t have an index, we can construct a hash table that is $B - 2$ pages² big on the records of $R$, fit it into memory, and then read in each record of $S$ and look it up in $R$’s hash table to see if we can find any matches on it. This is called Naive Hash Join. Its cost is $[R] + [S]$ I/Os.

That’s actually the best one we’ve done yet. It’s efficient, cheap, and simple. There’s a problem with this, however; this relies on $R$ being able to fit entirely into memory (specifically, having $R$ being $\leq B - 2$ pages big). And that’s often just not going to be possible.

To fix this, we repeatedly hash $R$ and $S$ into $B - 1$ partitions until we get partitions of $R$ small enough to load into memory ($\leq B - 2$ pages big). We should only hash the partitions if they are bigger than $B - 2$ pages (not all partitions will be the same size after hashing). After all of them are less than or equal to $B - 2$ pages big, we can we perform a Naive Hash Join on each of those subsections of $R$ and $S$ (i.e partition 1 of $R$ and partition 1 of $S$, partition 2 of $R$ and partition 2 of $S$, etc). This is called Grace Hash Join, and the I/O cost of this is: the cost of hashing plus the Naive Hash Join on the subsections. The cost of hashing can change based on how many times we need to repeatedly hash on how many partitions. The cost of hashing a partition $P$ includes the I/O’s we need to read all the pages in $P$ and the I/O’s we need to write all the resulting partitions after hashing partition $P$. The Naive Hash Join portion costs $[R] + [S]$.

Grace Hash is great, but it’s really sensitive to key skew, so you want to be careful when using this algorithm. Key skew is when we try to hash but many of the keys go into the same bucket. Key skew happens when many of the records have the same key. For example, if we’re hashing on the column which only has ”yes” as values, then we can keep hashing but they will all end up in the same bucket no matter which hash function we use.

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²We need one page for the current page in $S$ and one page to store output records. The other $B - 2$ pages can be used for the hash table.
There’s also times when it helps for us to sort \( R \) and \( S \) first, especially if we want our joined table to be sorted on some specific column. In those cases, what we do is first sort \( R \) and \( S \). Then...

1. we begin at the start of \( R \) and \( S \) and advance one or the other until we get to a match (if \( r_i < s_j \), advance \( R \); else if \( r_i > s_j \), advance \( S \) – the idea is to advance the lesser of the two until we get to a match).

2. Now, let’s assume we’ve gotten to a match. Let’s say this pair is \( r_i, s_j \). We mark this spot in \( S \) as \( marked(S) \) and check each subsequent record in \( S \) \( (s_j, s_{j+1}, s_{j+2}, \text{ etc}) \) until we find something that is not a match (i.e. read in all records in \( S \) that match to \( r_i \)).

3. Now, go to the next record in \( R \) and go back to the marked spot in \( S \) and begin again at step 1 (except instead of beginning at the start of \( R \) and the start of \( S \), do it at the indices we just indicated) – the idea is that because \( R \) and \( S \) are sorted, any match for any future record of \( R \) cannot be before the marked spot in \( S \), because this record \( r_{i+1} \geq r_i \) – if a match for \( r_i \) did not exist before \( marked(S) \), a match for \( r_{i+1} \) cannot possibly be before \( marked(S) \) either! So we scroll from the marked spot in \( S \) until we find a match for \( r_{i+1} \).

This is called **Sort-Merge Join** and the average I/O cost is: cost to sort \( R \) + cost to sort \( S \) + \((|R| + |S|)\) (though it is important to note that this is not the worst case!).\(^3\)

\(^3\)(in the worst case, if each record of \( R \) matches every record of \( S \), we could see the last term could be \(|R| + |S|\). That generally doesn’t happen, though).
Let’s take a look at an example. Let the table on the left be \( R \) and the table on the right be \( S \).

We will advance the pointer (the red arrow) on \( S \) because 28 < 31 until \( S \) gets to sid of 31. Then we will mark this record (the black arrow). In addition, we will output this match.

Then we will advance the pointer on \( S \) again and we get another match and output it.
We advance the pointer on $S$ again, but we do not get a match. We then reset $S$ to where we marked (the black arrow) and then advance $R$. When we advance $R$, we get another match so we output it.

We then advance $S$, we get another match so we output it.
do {
    if (!mark) {
        while (r < s) { advance r }
        while (r > s) { advance s }
        // mark start of block of S
        mark = s
    }
    if (r == s) {
        result = <r, s>
        advance s
        yield result
    } else {
        reset s to mark
        advance r
        mark = NULL
    }
}

7.1 An Important Refinement

An important refinement: You can combine the last sorting phase with the merging phase, provided you have enough room in memory to allocate a page for each run of [R] and for each run of [S]. The final merge pass is where you allocate a page for each run of R and each run of S. In this process, you save $2 \times ([R] + [S])$ I/Os.

To perform the optimization, we must

1. sort [R] and [S] (individually, using the full buffer pool for each of them) until you have them both “almost-sorted”; that is, for each table $T$ in $\{R, S\}$, keep merging runs of $T$ until you get to the penultimate step, where one more sorting pass would result in a sorted table $T$.

2. See how many runs of [R] and how many runs of [S] are left; sum them up. Allocate one page in memory for each run. If you have enough input buffers left to accommodate this (i.e. if $\text{runs}(R) + \text{runs}(S) \leq B - 1$), then you may use the optimization and you could then save $2 \times ([R] + [S])$ I/Os.

3. If you could not do the optimization described in the previous step, it means that $\text{runs}(R) + \text{runs}(S) \geq B$. In the ideal case, the optimization allows us to avoid doing an extra read of both $R$ and $S$, but this is not possible here, as we don’t have an available buffer for each run of $R$ and $S$. 
However, we’re not out of options yet! If we can’t avoid extra reads for both $R$ and $S$, perhaps we can avoid an extra read for at least one of them. Because we are trying to minimize I/Os, we would like to avoid the extra read for the larger table $R$. Could it help us to maybe completely sort just the smaller table $S$ and use that in an optimization?

It turns out that we can! If we completely sort $S$, there is, by definition, now only one sorted run for that table, and this would mean that we only need to allocate one page in our buffer for it. So, if $\text{runs}(R) + 1 \leq B - 1$, then we can allocate one buffer page for $S$ and for each run of $R$. Then, we can perform the SMJ optimization. Here, we have avoided the final sorting pass for $R$ by combining it with the join phase. So we have saved $2 \times [R]$ pages.

Sometimes, though, even this is not enough – if $\text{runs}(R) = B-1$, then we don’t have a spare buffer page for $S$. In this case, we’d like to still reduce as many I/Os as possible, so maybe perform the procedure described in the previous paragraph, but for the smaller table. That is, if $\text{runs}(S) + 1 \leq B - 1$, then sort $R$, reducing the number of sorted runs for it to 1 by definition, and then allocate one buffer page for $R$ and one for each run of $S$. Then, perform the optimization – this will enable us to avoid the final sorting pass for $S$ by combining it with the join phase, saving us $2 \times [S]$ I/Os.

If none of those conditions are met, though, we just won’t be able to optimize sort-merge join. And that’s alright. Sometimes, we just have to bite the bullet and accept that we can’t always cut corners.